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Long-time behavior for a nonlinear parabolic problem with variable exponents

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ABSTRACT

This paper addresses the question of the asymptotic behavior of solutions to the $p(x)$-Laplacian problem

$$u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + f(x,u) = g.$$

With general assumptions on $f(x,u)$ and the exponent $p(x)$, we prove the existence of global attractors in proper spaces. Then we consider the fractal dimension of global attractors for the problem. Under suitable conditions, we show that the problem admits an infinite-dimensional global attractor.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$. We consider the asymptotic behavior of solutions to the following $p(x)$-Laplacian equations:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) + f(x,u) = g, & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $g, u_0 \in L^2(\Omega)$, $p \in C(\overline{\Omega})$, with $2 \leq p(x) < \Lambda < \infty$, $x \in \overline{\Omega}$. Besides, $p(x)$ is log-Hölder continuous, i.e., there exists a constant C such that

$$|p(x) - p(y)| \leq -\frac{C}{\log|x-y|}, \quad \text{for every } x, y \in \Omega \text{ with } |x-y| < \frac{1}{2}. \tag{1.2}$$

We assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping and that there exist positive constants l, k, c_1, c_2 such that

$$(f(x,s_1) - f(x,s_2))(s_1 - s_2) \geq -l|s_1 - s_2|^2, \quad \text{for any } x \in \Omega \text{ and } s_1, s_2 \in R, \tag{1.3}$$

$$c_2|s|^{q(x)} - k \leq f(x,s)s \leq c_1|s|^{q(x)} + k, \quad \text{for any } x \in \Omega \text{ and } s \in R, \tag{1.4}$$

where $q(x) \in C(\overline{\Omega})$, with

$$2 \leq q^- = \inf_{x\in\Omega} q(x), \qquad q^+ = \sup_{x\in\Omega} q(x).$$

In recent years, parabolic and elliptic problems with variable exponents have been studied extensively by many authors; see, for example, [1–13] and the references therein. The wide study of such problems is motivated by various

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applications related to electrorheological fluids (an important class of non-Newtonian fluids) [1,14,15], image processing [6], elasticity [16], and also mathematical biology [7].

The $p(x)$-Laplacian problem (1.1) can be viewed as a generalization of p-Laplacian equations

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) + f(x,u) = g & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

The long-time behavior of solutions to p-Laplacian equations has been studied widely; see, for example, [17–24,12]. The global attractors have been obtained in various spaces and in different contexts. The fractal dimension of global attractors for p-Laplacian equations has also been considered; see [25].

However, until now, few results have been obtained concerning the long-time behavior of solutions for $p(x)$-Laplacian equations. In [9], the author studied the existence of global attractors for problem (1.1) with $g = 0$ and $f(x,u) = -B(u)$, with B a globally Lipschitz map. A global attractor was obtained in $L^2(\Omega)$. In [10], the upper semicontinuity of global attractors for $p(x)$-Laplacian equations was studied.

In this paper, we shall consider the long-time behavior of solutions to problem (1.1) with general $g \in L^2(\Omega)$. First, we prove that the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (1.1) possesses a global attractor in $L^{q(x)}(\Omega)$. The main idea is to obtain asymptotic compactness of the semigroup in $L^{q(x)}(\Omega)$ by the compactness of the semigroup in $L^2(\Omega)$. When $N = 1$ or 2, the existence of global attractors in $L^{q(x)}(\Omega)$ for the problem can be obtained easily by means of compact imbeddings of Sobolev spaces. We are mainly focused on the case $N \geq 3$ in this paper. After we obtain the global attractor, we consider its fractal dimension. Under proper assumptions, we show that the lower bound for the fractal dimension of the global attractor can be arbitrarily large; that is, problem (1.1) admits a global attractor with infinite fractal dimension.

Generally, it is a bit tricky to estimate lower bounds for the dimension of global attractors for degenerate problems. The general method to estimate lower bounds for the dimension of global attractors is the unstable manifold method [23], which needs the differentiability of the semigroup. Since the semigroups associated with degenerate problems are usually not differentiable, the unstable manifold method is no longer suitable. Recently, in [25,26], the authors developed a method to estimate the lower bounds for the dimension of global attractors for some degenerate problems. However, the method is highly based on the regularity of solutions and a scaling technique, and thus is not suitable for $p(x)$-Laplacian equations. In this paper, we use the frame we introduced in [12] to consider the dimension of the global attractor.

To consider problems with variable exponents, one needs the basic theory of spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For the convenience of readers, let us review them briefly here. The details and more properties of variable-exponent Lebesgue–Sobolev spaces can be found in [27–30], and the more recent monograph [31].

Let $p \in C(\overline{\Omega})$. When $p^- > 1$, one can introduce the variable-exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R};\ u \text{ is measurable and } \int_\Omega |u|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$\|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1 \right\}.$$

Thanks to results in [29], the following inequality holds:

$$\min\{\|u\|_{p(x)}^{p^+}, \|u\|_{p(x)}^{p^-}\} \leq \int_\Omega |u|^{p(x)} dx \leq \max\{\|u\|_{p(x)}^{p^+}, \|u\|_{p(x)}^{p^-}\}.$$

Moreover, let $r_i \in C(\overline{\Omega})$, with $r_i^- > 1$, $i = 1, 2$. Then, if $r_1(x) \leq r_2(x)$ for any $x \in \Omega$, the imbedding $L^{r_2(x)}(\Omega) \hookrightarrow L^{r_1(x)}(\Omega)$ is continuous, the norm of the imbedding does not exceed $|\Omega| + 1$. As $p^- > 1$, and the space is a reflexive Banach space with dual $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Besides, for any $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality:

$$\int_\Omega |uv| dx \leq \left(\frac{1}{p^-} + \frac{1}{(p^-)'} \right) \|u\|_{p(x)} \|v\|_{p'(x)}.$$

For positive integer k, the generalized Lebesgue–Sobolev space is defined as

$$W^{k,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \leq k\},$$

endowed with the norm

$$\|u\|_{W^{k,p(x)}} = \sum_{\alpha \leq k} \|D^\alpha u\|_{p(x)}.$$

Such spaces are separable and reflexive Banach spaces.

Under assumption (1.2), the smooth functions are dense in Sobolev spaces with variable exponents, and we can define $W_0^{k,p(x)}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ in $W^{k,p(x)}(\Omega)$ with respect to the norm $\|\cdot\|_{W^{k,p(x)}}$; see [31,29]. For $u \in W_0^{1,p(x)}(\Omega)$, the Poincaré- type inequality holds, i.e.,

$$\|u\|_{p(x)} \leq C\|\nabla u\|_{p(x)},$$

where the positive constant C depends on p and Ω. So $\|\nabla u\|_{p(x)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. Note that the dense result is generally not true without additional assumptions on the exponent $p(x)$;, see [29].

Now, we state our main results as follows.

Theorem 1.1. *Assume that $u_0, g \in L^2(\Omega)$, $p \in C(\overline{\Omega})$, with $2 \leq p^- \leq p^+ < \infty$. Let $p(x)$ satisfy (1.2) and f satisfy assumptions (1.3) and (1.4), with $2 \leq q^- \leq q^+ < \infty$. Then the semigroup associated with problem (1.1) possesses a global attractor $\mathscr{A}$ in $L^{q(x)}(\Omega)$; i.e., $\mathscr{A}$ is compact, invariant in $L^{q(x)}(\Omega)$, and attracts every bounded subset of $L^2(\Omega)$ in the norm topology of $L^{q(x)}(\Omega)$.*

In the following, in addition to (1.3) and (1.4), we assume that

$$f(x,u) = f_1(x,u) - |u|^{s(x)-2}u, \tag{1.6}$$

with $f_1(x,u)$ being odd with respect to u, and $s \in C(\overline{\Omega})$, $2 \leq s^- \leq s^+ < p^-$. Moreover, we assume that

$$\lim_{|t|\to 0} \frac{|f_1(x,t)t|}{|t|^{\alpha_0}} = 0, \quad \text{uniformly in } x \tag{1.7}$$

holds for some $\alpha_0 > s^+$. Then we have the following result.

Theorem 1.2. *Let $u_0 \in L^2(\Omega)$, $g = 0$, $p \in C(\overline{\Omega})$, with $2 < p^- \leq p^+ < \infty$. Assume that $p(x)$ satisfies condition (1.2) and f satisfies assumptions (1.3), (1.4), (1.6), and (1.7). Then the semigroup associated with problem (1.1) admits a symmetric global attractor in $L^2(\Omega)$, the fractal dimension of which is infinite.*

Remark 1.1. A typical example of $f(x,u)$ is $|u|^{r-2}u - |u|^{s-2}u$, with $2 \leq s < p^-$, $s < r$.

The remaining parts of the paper are arranged as follows. We prove the existence of global attractors in $L^2(\Omega)$ and $L^{q(x)}(\Omega)$ respectively in Sections 2 and 3. Then, in Section 4, we consider the fractal dimension of the global attractor.

For convenience, we denote by $|E|$ the Lebesgue measure of the set E, and denote by C any positive constant, which may be different even in the same line. Besides, we use $\Omega(|u| \geq M)$ to denote the set $\{x \in \Omega : |u(x)| \geq M\}$ hereafter.

2. Global attractors in $L^2(\Omega)$

In this section, we provide the existence results for problem (1.1), and then we show the existence of a global attractor in $L^2(\Omega)$.

Denote

$$X = \{u : u \in L^{p^-}(0,T; W_0^{1,p(x)}(\Omega)) \cap L^\infty(0,T; L^2(\Omega)) \cap L^{q(x)}(\Omega \times (0,T)) \text{ with } \nabla u \in L^{p(x)}(\Omega \times (0,T))\}.$$

Definition 2.1. A solution of problem (1.1) is a function $u \in X$ such that

$$\int_0^t \int_\Omega (-u\varphi_t + |\nabla u|^{p(x)-2}\nabla u \nabla\varphi + f(x,u)\varphi)dxd\xi = \int_0^t \int_\Omega g\varphi dxd\xi - \int_\Omega u\varphi dx\,|_0^t$$

holds for any $t \leq T$ and all $\varphi \in X$ with $\varphi_t \in X^*$, where X^* is the dual space of X.

Theorem 2.1. *Let $u_0, g \in L^2(\Omega)$, $p \in C(\overline{\Omega})$, with $2 \leq p^- \leq p^+ < \infty$. Let p satisfy (1.2), and let f satisfy assumptions (1.3) and (1.4), with $2 \leq q^- \leq q^+ < \infty$. Then problem (1.1) admits a unique solution $u \in C([0,T]; L^2(\Omega))$. Moreover, the mapping $u_0 \to u(t)$ is continuous in $L^2(\Omega)$.*

Proof. The existence result above was obtained by Antontsev and Shmarev [2] using Galerkin methods. Actually, they considered a more general model. One can also obtain the result using variational methods, similarly to [11].

Let u and v be two different solutions of problem (1.1), Then $w = u - v$ satisfies the following equations:

$$\begin{cases} w_t - (\mathrm{div}(|\nabla u|^{p(x)-2}\nabla u) - \mathrm{div}(|\nabla v|^{p(x)-2}\nabla v)) + f(x,u) - f(x,v) = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ w = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ w(x,0) = 0 & \text{in } \Omega. \end{cases}$$

Taking w as a test function, and using assumption (1.3), we have

$$\frac{d}{dt}\int_\Omega |w|^2 dx \leq l \int_\Omega |w|^2 dx.$$

Then it is not difficult to obtain that $w = 0$. Thus the solution is unique. The continuity of the mapping $u_0 \to u(t)$ can be obtained similarly. $\square$

From Theorem 2.1, the solution of problem (1.1) generates a semigroup $\{S(t)\}_{t\geq 0}$ in $L^2(\Omega)$. Next, we show that the semigroup possesses a global attractor in $L^2(\Omega)$.

Theorem 2.2. *Under the assumptions of Theorem 2.1, the semigroup $\{S(t)\}_{t\geq 0}$ associated with problem (1.1) admits an absorbing set in $W_0^{1,p(x)}(\Omega)\cap L^{q(x)}(\Omega)$; i.e., there is a bounded set $B_0 \subset W_0^{1,p(x)}(\Omega)\cap L^{q(x)}(\Omega)$ such that, for any bounded set B in $L^2(\Omega)$, there exists a $T_0 > 0$ such that*

$$S(t)B \subset B_0 \quad \text{for any } t \geq T_0,$$

where T_0 depends only on B.

Proof. Multiplying problem (1.1) by u, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2dx + \int_\Omega |\nabla u|^{p(x)}dx + \int_\Omega f(x,u)udx = \int_\Omega gudx.$$

Note that assumption (1.4) implies that

$$\int_\Omega f(x,u)udx \geq c_2\int_\Omega |u|^{q(x)}dx - k|\Omega|.$$

Using Young's inequality, we obtain

$$\frac{d}{dt}\int_\Omega |u|^2dx + \int_\Omega (|\nabla u|^{p(x)} + |u|^{q(x)})dx \leq C_\varepsilon\int_\Omega |g|^2dx + \varepsilon\int_\Omega |u|^2dx + C|\Omega|. \tag{2.1}$$

Again, from Young's inequality,

$$\int_\Omega |u|^2dx \leq \int_\Omega \frac{2}{q(x)}|u|^{q(x)}dx + \int_\Omega \frac{q(x)-2}{q(x)}\cdot 1^{\frac{q(x)}{q(x)-2}}dx \leq \int_\Omega \frac{2}{q(x)}|u|^{q(x)}dx + C|\Omega|,$$

we have

$$\int_\Omega |u|^2dx \leq \int_\Omega |u|^{q(x)}dx + C|\Omega|.$$

Taking the inequality above into (2.1), we get

$$\frac{d}{dt}\int_\Omega |u|^2dx + \int_\Omega (|\nabla u|^{p(x)} + |u|^{q(x)})dx \leq C_\varepsilon\int_\Omega |g|^2dx + C|\Omega|, \tag{2.2}$$

and also

$$\frac{d}{dt}\|u\|_2^2 + C\|u\|_2^2 \leq C,$$

which implies that

$$\|u(t)\|_2 \leq C \quad \text{for any } t \geq t_0.$$

So the semigroup has an absorbing set in $L^2(\Omega)$. Integrating (2.2) over $[t, t+1]$, $t \geq t_0$, we obtain

$$\int_t^{t+1}\int_\Omega (|\nabla u|^{p(x)} + |u|^{q(x)})dxds \leq C_\varepsilon\|g\|_2^2 + \|u(t)\|_2^2 + C|\Omega| \leq C, \quad \text{for } t \geq t_0. \tag{2.3}$$

Now, multiplying problem (1.1) by u_t, we get

$$\int_\Omega |u_t|^2dx + \frac{d}{dt}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \frac{d}{dt}\int_\Omega F(x,u)dx \leq C\|g\|_2^2, \tag{2.4}$$

where $F(x,s)$ is the primitive function of $f(x,s)$; i.e., $F(x,s) = \int_0^s f(x,\tau)d\tau$, $x\in\Omega$. From assumption (1.4), we have

$$C_2|u|^{q(x)} - C \leq F(x,u) \leq C_1|u|^{q(x)} + C. \tag{2.5}$$

Integrating (2.4) over $[s, t+1]$, $t_0 \leq t < s < t+1$, yields

$$\int_\Omega \frac{1}{p(x)}|\nabla u(t+1)|^{p(x)}dx + \int_\Omega F(x,u(t+1))dx \leq C\|g\|_2^2 + \int_\Omega \frac{1}{p(x)}|\nabla u(s)|^{p(x)}dx + \int_\Omega F(x,u(s))dx.$$

Integrating the above inequality with respect to s between t and $t+1$, we obtain

$$\begin{aligned}&\int_\Omega \frac{1}{p(x)}|\nabla u(t+1)|^{p(x)}dx+\int_\Omega F(x,u(t+1))dx\\&\quad\le C\|g\|_2^2+\int_t^{t+1}\int_\Omega \frac{1}{p(x)}|\nabla u(s)|^{p(x)}dxds+\int_t^{t+1}\int_\Omega F(x,u(s))dxds.\end{aligned}$$

Combining the assumptions on $p(x)$ and (2.3) and (2.5), we get

$$\begin{aligned}\int_\Omega |\nabla u(t+1)|^{p(x)}dx+\int_\Omega |u(t+1)|^{q(x)}dx&\le C\int_\Omega \frac{1}{p(x)}|\nabla u(t+1)|^{p(x)}dx+C\int_\Omega F(x,u(t+1))dx+C\\&\le C(\|g\|_2,|\Omega|).\end{aligned}$$

We then conclude that

$$\|\nabla u\|_{p(x)}+\|u\|_{q(x)}\le C,\quad \text{for all } t\ge t_0+1. \tag{2.6}$$

The proof is completed. □

Remark 2.1. Thanks to the compact imbedding results in [29] and Theorem 6.2 in [32], we are now in a position to obtain the global attractor in $L^2(\Omega)$.

Corollary 2.1. *Under the assumptions in Theorem 2.1, the semigroup associated with problem (1.1) possesses a global attractor $\mathscr{A}$ in $L^2(\Omega)$; i.e., $\mathscr{A}$ is compact, invariant in $L^2(\Omega)$, and attracts every bounded subset of $L^2(\Omega)$ in the norm topology.*

3. Global attractors in $L^{q(x)}(\Omega)$

In this section, we shall prove the existence of a global attractor in $L^{q(x)}(\Omega)$. The main idea is to use the asymptotic a priori estimate for the unbounded part of modular $|u|$ to get the compactness of the semigroup in $L^{q(x)}(\Omega)$ by the compactness of the semigroup in $L^2(\Omega)$. We first provide some abstract results, which are modified from [13].

Lemma 3.1. *Let $q\in C(\overline{\Omega})$, with $1\le r\le q^-\le q^+<\infty$. For any $\varepsilon>0$, a bounded subset B of $L^{q(x)}(\Omega)$ has a finite ε-net in $L^{q(x)}(\Omega)$ if there exists a positive constant $M=M(\varepsilon)$ such that*

(i) *B has finite $(\frac{1}{2})^{1/r}(3M)^{(r-q^+)/r}\varepsilon^{q^+/r}$-net in $L^r(\Omega)$,*
(ii) *$\int_{\Omega(|u|\ge M)}|u|^{q(x)}dx<2^{-(q^++3)}\varepsilon^{q^+}$ for any $u\in B$.*

Proof. From assumption (i), there exist $u_i\in B$, $1\le i\le k$, such that, for each $u\in B$, there is a u_i satisfying

$$\int_\Omega |u-u_i|^r dx<\frac{1}{2}(3M)^{(r-q+)}\varepsilon^{q^+}.$$

Without loss of generality, we assume that $M>1$. Note that

$$\begin{aligned}\int_\Omega |u-u_i|^{q(x)}dx&\le \int_{\Omega(|u-u_i|\ge 3M)}|u-u_i|^{q(x)}dx+\int_{\Omega(|u-u_i|\le 3M)}|u-u_i|^{q(x)}dx\\&\le \int_{\Omega(|u-u_i|\ge 3M)}|u-u_i|^{q(x)}dx+(3M)^{q^+-r}\int_{\Omega(|u-u_i|\le 3M)}|u-u_i|^r dx\\&<\int_{\Omega(|u-u_i|\ge 3M)}|u-u_i|^{q(x)}dx+\frac{1}{2}\varepsilon^{q^+}.\end{aligned}\tag{3.1}$$

Denote

$$\begin{aligned}\Omega_1&=\Omega\left(|u|\ge\frac{3M}{2}\right)\cap\Omega\left(|u_i|\le\frac{3M}{2}\right),\\ \Omega_2&=\Omega\left(|u|\le\frac{3M}{2}\right)\cap\Omega\left(|u_i|\ge\frac{3M}{2}\right),\\ \Omega_3&=\Omega\left(|u|\ge\frac{3M}{2}\right)\cap\Omega\left(|u_i|\ge\frac{3M}{2}\right).\end{aligned}$$

By the inequality

$$|u-u_i|^{q(x)}\le 2^{q^+}(|u|^{q(x)}+|u_i|^{q(x)}),$$

we deduce that

$$\begin{aligned}\int_{\Omega(|u-u_i|\geq 3M)}|u-u_i|^{q(x)}dx &\leq \int_{\Omega_1}|u-u_i|^{q(x)}dx+\int_{\Omega_2}|u-u_i|^{q(x)}dx+\int_{\Omega_3}|u-u_i|^{q(x)}dx\\ &\leq 2^{q^+}\left(\int_{\Omega_1}|u|^{q(x)}dx+\int_{\Omega_2}|u_i|^{q(x)}dx\right)+2^{q^+}\int_{\Omega_3}(|u|^{q(x)}+|u_i|^{q(x)})dx\\ &\leq 2^{q^++1}\left(\int_{\Omega(|u|\geq M)}|u|^{q(x)}dx+\int_{\Omega(|u_i|\geq M)}|u_i|^{q(x)}dx\right)\\ &\leq 2^{q^++2}2^{-q^+-3}\varepsilon^{q^+}=\frac{1}{2}\varepsilon^{q^+}.\end{aligned} \tag{3.2}$$

Combining (3.1) and (3.2), we conclude that

$$\min\{\|u-u_i\|_{q(x)}^{q^+},\|u-u_i\|_{q(x)}^{q^-}\}\leq\int_{\Omega}|u-u_i|^{q(x)}dx<\varepsilon^{q^+},$$

which implies that $\|u-u_i\|_{q(x)}<\varepsilon$. Hence, the proof of the lemma is completed. □

To state the second lemma, we recall the definition of the Kuraktowski measure of noncompactness. Interested readers can refer to [33,13] for details and some properties of it.

Definition 3.1. Let X be a complete metric space, and let A be a bounded subset of X. The Kuraktowski measure of noncompactness $\kappa(A)$ of A is defined as

$$\kappa(A)=\inf\{\delta>0|A \text{ has finite open cover of sets with diameter } <\delta\}.$$

With Lemma 3.1, we prove the following result.

Lemma 3.2. *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on $L^r(\Omega)$, $r\geq 1$, which possesses a global attractor in the space. Let $q(x)\in C(\overline{\Omega})$, with $r\leq q^-\leq q^+<\infty$. If, for any $\varepsilon>0$, and any bounded subset $B\subset L^{q(x)}(\Omega)$, there exist positive constants $M=M(\varepsilon)$ and $T=T(\varepsilon,B)$, such that*

$$\int_{\Omega(|S(t)u_0|\geq M)}|S(t)u_0|^{q(x)}dx<\varepsilon\quad\text{for any } u_0\in B \text{ and } t\geq T, \tag{3.3}$$

then $\{S(t)\}_{t\geq 0}$ is ω-limit compact in $L^{q(x)}(\Omega)$; that is, for every bounded subset B of $L^{q(x)}(\Omega)$, and for any $\varepsilon>0$, there exists a $T=T(B,\varepsilon)$ such that

$$\kappa(\cup_{t\geq T}S(t)B)<\varepsilon.$$

Proof. It is enough to prove that, for any bounded subset B of $L^{q(x)}(\Omega)$, and for any $\varepsilon>0$, there exists a $T=T(\varepsilon,B)$ such that $\cup_{t\geq T}S(t)B$ has a finite ε-net in $L^{q(x)}(\Omega)$. From assumption (3.3) and Lemma 3.1, it is enough to prove that $\cup_{t\geq T}S(t)B$ has a finite ε-net in $L^r(\Omega)$.

From the imbedding $L^{q(x)}(\Omega)\hookrightarrow L^r(\Omega)$, B is also bounded in $L^r(\Omega)$. Thus there exists a $T_0=T_0(\varepsilon,B)$ such that $\cup_{t\geq T_0}S(t)B\subset\mathcal{O}(\mathscr{A},\frac{\varepsilon}{2})$, where $\mathcal{O}(\mathscr{A},\frac{\varepsilon}{2})$ denotes the $\frac{\varepsilon}{2}$ neighborhood of $\mathscr{A}$. Since $\mathscr{A}$ is compact in $L^r(\Omega)$, it has a finite $\frac{\varepsilon}{2}$-net. So $\cup_{t\geq T_0}S(t)B$ has a finite ε-net in $L^r(\Omega)$. Thus, combining (3.3) and Lemma 3.1, we get the results of Lemma 3.2. □

Let $q(x)\in C(\overline{\Omega})$ with $1<r\leq q^-\leq q^+$. Note that $L^{q^+}(\Omega)\subset L^{q(x)}(\Omega)\subset L^{q^-}(\Omega)\subset L^r(\Omega)$, and that $L^{r'}(\Omega)$ is dense in $L^{q'(x)}(\Omega)$, where $\frac{1}{q(x)}+\frac{1}{q'(x)}=1$. Thanks to Corollary 3.6 in [13], a continuous semigroup in $L^r(\Omega)$ is norm to weak continuous in $L^{q(x)}(\Omega)$. Thus, from Theorem 4.2 in [13] and Lemma 3.2 above, we have the following result.

Proposition 3.1. *Let $\{S(t)\}_{t\geq 0}$ be a semigroup on $L^r(\Omega)$, $r>1$, and let it possess a global attractor in it. Then $\{S(t)\}_{t\geq 0}$ has a global attractor in $L^{q(x)}(\Omega)$, with $r\leq q^-\leq q^+<\infty$, if*

(i) *$\{S(t)\}_{t\geq 0}$ has a bounded absorbing set B_0 in $L^{q(x)}(\Omega)$, and*
(ii) *for any $\varepsilon>0$, and any bounded subset $B\subset L^{q(x)}(\Omega)$, there exist positive constants $M=M(\varepsilon)$ and $T=T(\varepsilon,B)$, such that*

$$\int_{\Omega(|S(t)u_0|\geq M)}|S(t)u_0|^{q(x)}<\varepsilon\quad\text{for any } u_0\in B \text{ and } t\geq T.$$

Now, we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. According to the proposition above and (2.6), we need only to prove that condition (ii) holds for the semigroup associated with problem (1.1); that is, that, for any $B_0 \subset L^2(\Omega)$, there exist $T_0 = T_0(\varepsilon, B_0)$ and $M_0 = M_0(\varepsilon)$ such that

$$\int_{\Omega(|S(t)u_0| \geq M_0)} |S(t)u_0|^{q(x)} dx < \varepsilon \quad \text{for any } u_0 \in B \text{ and } t \geq T_0. \tag{3.4}$$

The proof is modified from Theorem 5.10 in [13].

It is obvious that, for any $\varepsilon > 0$, there is a $\delta > 0$ such that, if $E \subset \Omega$ with $|E| < \delta$, then

$$\int_E |g(x)|^2 dx < \varepsilon.$$

Thanks to Lemmas 5.2 and 5.6 in [13] and Corollary 2.1 above, there exist $T_1 = T_1(\varepsilon, B_0)$ and $M_1 = M_1(\varepsilon)$ such that

$$|\Omega(|S(t)u_0| \geq M_1)| \leq \min\{\varepsilon, \delta\},$$
$$\int_{\Omega(|S(t)u_0| \geq M_1)} |S(t)u_0|^2 dx \leq 8\varepsilon \tag{3.5}$$

hold for any $u_0 \in B_0, t \geq T_1$ (see also (5.26) and (5.27) in [13]). Besides, from assumption (1.4), $f(x,s) \geq 0$ for $s > \max\{(\frac{k}{c_2})^{1/q^-}, (\frac{k}{c_2})^{1/q^+}\}$. Let $M = \max\{M_1, (\frac{k}{c_2})^{1/q^-}, (\frac{k}{c_2})^{1/q^+}\}$, $t \geq T_1$. Define

$$(u-M)_+ = \begin{cases} u - M, & u > M \\ 0, & u \leq M. \end{cases}$$

Multiplying (1.1) by $(u-M)_+$, and integrating over Ω, we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u-M)_+^2 dx + \int_\Omega |\nabla(u-M)_+|^{p(x)} dx + \int_\Omega f(x,u)(u-M)_+ dx = \int_\Omega g(u-M)_+ dx.$$

Integrating between t and $t+1$, we get

$$\begin{aligned}
&\int_t^{t+1}\int_{\Omega(u\geq M)} |\nabla(u-M)_+|^{p(x)} dxds + \int_t^{t+1}\int_{\Omega(u\geq M)} f(x,u)(u-M)_+ dxds \\
&\quad \leq \frac{1}{2}\int_{\Omega(u\geq M)} (u-M)_+^2 dx + \frac{1}{2}\int_t^{t+1}\int_{\Omega(u\geq M)} (|g|^2 + (u-M)_+^2) dxds.
\end{aligned}$$

We then conclude from (3.5) that

$$\int_t^{t+1}\int_{\Omega(u\geq M)} |\nabla(u-M)_+|^{p(x)} dxds + \int_t^{t+1}\int_{\Omega(u\geq M)} f(x,u)(u-M)_+ dxds \leq C\varepsilon, \tag{3.6}$$

where C is independent of ε, M. Using assumption (1.4), we deduce that

$$\begin{aligned}
c_2\int_t^{t+1}\int_{\Omega(u\geq 2M)} |u|^{q(x)} dxds &\leq \int_t^{t+1}\int_{\Omega(u\geq 2M)} f(x,u)u dxds + k\varepsilon \\
&\leq 2\int_t^{t+1}\int_{\Omega(u\geq 2M)} f(x,u)(u-M)_+ dxds + k\varepsilon \\
&\leq 2\int_t^{t+1}\int_{\Omega(u\geq M)} f(x,u)(u-M)_+ dxds + k\varepsilon \\
&\leq C\varepsilon.
\end{aligned} \tag{3.7}$$

Now we multiply (1.1) by $((u-2M)_+)_t$ to obtain that

$$\frac{d}{dt}\int_{\Omega(u\geq 2M)} \frac{1}{p(x)}|\nabla(u-2M)_+|^{p(x)} dx + \frac{d}{dt}\int_{\Omega(u\geq 2M)} F(x,u) dx \leq \int_{\Omega(u\geq 2M)} |g|^2 dx. \tag{3.8}$$

Integrating (3.8) from s to $t+1$, $(s \in (t, t+1))$, yields

$$\begin{aligned}
&\int_{\Omega(u\geq 2M)} \frac{1}{p(x)}|\nabla(u(t+1)-2M)_+|^{p(x)} dx + \int_{\Omega(u\geq 2M)} F(x, u(t+1)) dx \\
&\quad \leq \int_{\Omega(u\geq 2M)} \frac{1}{p(x)}|\nabla(u(s)-2M)_+|^{p(x)} dx + \int_{\Omega(u\geq 2M)} F(x, u(s)) dx + C\varepsilon.
\end{aligned}$$

Integrating the above inequality with respect to s over $[t, t+1]$, and combining (2.5), (3.6), and (3.7) we obtain

$$\begin{aligned}\int_{\Omega(u\geq 2M)} |u(t+1)|^{q(x)}dx &\leq C\int_{\Omega(u\geq 2M)} F(x,u(t+1))dx + C\varepsilon\\ &\leq C\int_t^{t+1}\int_{\Omega(u\geq 2M)}\left(\frac{1}{p(x)}|\nabla(u-2M)_+|^{p(x)} + F(x,u)\right)dxds + C\varepsilon\\ &\leq C\int_t^{t+1}\int_{\Omega(u\geq 2M)}\left(\frac{1}{p^-}|\nabla(u-2M)_+|^{p(x)} + F(x,u)\right)dxds + C\varepsilon\\ &\leq C\varepsilon.\end{aligned}$$

Taking $(u+M)_-$ and $((u+2M)_-)_t$ as test functions, and performing the same calculations as above, we can obtain

$$\int_{\Omega(u\leq -2M)} |u(t+1)|^{q(x)}dx \leq C\varepsilon. \tag{3.9}$$

Taking $M_0 = 2M$, $T_0 = T_1 + 1$, we get (3.4). Thus the proof of Theorem 1.1 is completed. □

Remark 3.1. Instead of (1.3), if we assume that $f : \Omega\times\mathbb{R}\to\mathbb{R}$ is of class C^1 in u with $f'(x,u) > -l$, $a.e.x\in\Omega$ for some positive constant l, then it is possible to obtain a global attractor in $W_0^{1,p(x)}(\Omega)$.

4. Infinite-dimensional global attractors

In this section, we consider the fractal dimension of the global attractor. We shall prove Theorem 1.2. We first prove that the Z_2 index of the global attractor is infinite. Then, by the Mañé projection theorem [34,35], we obtain the infinite dimensionality of the global attractor.

Let V be a Banach space. Define $\Sigma = \{A\subset V | A \text{ closed}, A = -A\}$ to be the class of closed symmetric subsets of V. For $A\in\Sigma$, the Z_2 index $\gamma(A)$ of A is defined as follows:

$$\gamma(A) = \begin{cases}\inf\{m : \exists h\in C^0(A;\mathbb{R}^m\setminus\{0\}), h(-u) = -h(u)\},\\ \infty \quad \text{if } \{\cdots\} = \emptyset, \text{ in particular, if } 0\in A,\\ 0 \quad A = \emptyset.\end{cases}$$

The Z_2 index defined above has the following properties, see [36].

Lemma 4.1. *A Z_2 index defined on Σ satisfies*

(1) $\gamma(A) = 0 \Leftrightarrow A = \emptyset$.
(2) *If $A\subset B$, then $\gamma(A)\leq\gamma(B)$, for any $A, B\subset\Sigma$.*
(3) $\gamma(A\cup B)\leq\gamma(A)+\gamma(B)$ *for any $A, B\subset\Sigma$.*
(4) *If $A\in\Sigma$ is a compact set, then $\exists\delta > 0$ such that $\gamma(\overline{N_\delta(A)}) = \gamma(A)$, $N_\delta(A)$ is a symmetric δ-neighborhood of A.*
(5) $\gamma(A)\leq\gamma(\overline{h(A)})$, $\forall A\in\Sigma$, *$h : V\to V$ is odd and continuous.*

Before providing the proof of Theorem 1.2, we give the following lemma.

Lemma 4.2. *Let $\{S(t)\}_{t\geq 0}$ be an odd semigroup on a complete metric space X, which possess a symmetric global attractor $\mathscr{A}$. If there exists a symmetric set B with $\gamma(B)\geq m$, $m\in[1,\infty)$, such that $\omega(B) = \cap_{s\geq 0}\overline{\cup_{t\geq s} S(t)B}\subset\mathscr{A}\setminus\{0\}$, then $\gamma(\mathscr{A}\setminus\mathscr{O}(0))\geq m$ for some neighborhood $\mathscr{O}(0)$ of 0.*

Proof. Since $\omega(B)\subset\mathscr{A}\setminus\{0\}$ and $\omega(B)$ is closed and compact, there exist open neighborhoods of 0 and $\omega(B)$, denoted respectively as $\mathscr{O}(0)$ and $\mathscr{N}(\omega(B))$, such that

$$\mathscr{O}(0)\cap\mathscr{N}(\omega(B)) = \emptyset.$$

Since $S(t)B\subset\mathscr{N}(\omega(B))$ for t large enough, we have $S(t)B\subset\mathscr{N}(\mathscr{A})$ for t large enough. Therefore, there exists T_B such that, for $t > T_B$,

$$S(t)B\subset\mathscr{N}(\omega(B))\subset\mathscr{N}(\mathscr{A})\setminus\mathscr{O}(0)\subset\mathscr{N}(\mathscr{A}\setminus\mathscr{O}(0)).$$

Note that $\mathscr{A}\setminus\mathscr{O}(0)$ is compact. Choosing a proper neighborhood $\mathscr{N}(\mathscr{A}\setminus\mathscr{O}(0))$, by (4) in Lemma 4.1, we have

$$\gamma(\mathscr{A}\setminus\mathscr{O}(0)) = \gamma(\overline{\mathscr{N}(\mathscr{A}\setminus\mathscr{O}(0))})\geq\gamma(\overline{S(t)B})\geq\gamma(B)\geq m, t \text{ large enough}.$$

The proof is completed. □

Proof of Theorem 1.2. From the assumptions of the theorem, it is easy to obtain that the solution semigroup of problem (1.1) is odd. Let B be a symmetric absorbing set. Then the symmetry of the global attractor follows from the fact that

$$\mathscr{A} = \omega(B) = \cap_{s\geq 0}\overline{\cup_{t\geq s} S(t)B}.$$

Next, we prove the following claim. □

Claim. *For any $m \in \mathbb{N}$, there exists a neighborhood $\mathcal{O}(0)$ of 0 such that the Z_2 index of the set $\mathscr{A}\setminus\mathcal{O}(0)$ satisfies $\gamma(\mathscr{A}\setminus\mathcal{O}(0)) \geq m$.*

Proof. According to Lemma 4.2, for any integer $m > 0$, we need only to find a symmetric set B_m with $\gamma(B_m) \geq m$, $\omega(B_m) = \cap_{s\geq 0}\overline{\cup_{t\geq s} S(t)B_m} \subset \mathscr{A} \setminus \{0\}$.

Consider the energy function

$$\Phi(u) = \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} - \frac{1}{s(x)}|u|^{s(x)} + F_1(x,u)\right) dx,$$

where $F_1(x,u) = \int_0^u f_1(x,s)ds$. Let $u(t)$ be the solution of problem (1.1). Multiplying (1.1) by u_t, we have

$$\|u_t\|_2^2 + \frac{d}{dt}\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx - \frac{d}{dt}\int_\Omega \frac{1}{s(x)}|u|^{s(x)}dx + \frac{d}{dt}\int_\Omega F_1(x,u)dx = 0.$$

Hence,

$$\frac{d}{dt}(\Phi(u)) = -\|u_t\|_2^2.$$

Thus, for each $u_0 \in W_0^{1,p^+}(\Omega) \cap L^{q^+}(\Omega)$, the function $t \to \Phi(u(t))$ is nonincreasing.

For any integer $m > 0$, let V_m be an m-dimensional subspace of $W_0^{1,p^+}(\Omega) \cap L^{q^+}(\Omega)$. Setting $A_m = \{u \in V_m : \|\nabla u\|_{p^+} + \|u\|_{q^+} = 1\}$, then A_m is compact in $W_0^{1,p^+}(\Omega) \cap L^{q^+}(\Omega)$ and $L^{s(x)}(\Omega) \cap L^{\alpha_0}(\Omega)$. So, there exist $\delta_1 > 0$, $\delta_2 > 0$ such that

$$\inf_{u\in A_m} \|u\|_{s(x)} = \delta_1, \qquad \sup_{u\in A_m} \|u\|_{\alpha_0} = \delta_2.$$

From assumptions (1.4) and (1.7) we have $|F_1(x,u)| < C(|u|^{q^+} + |u|^{\alpha_0})$ for some positive constant C. Setting $\epsilon A_m = \{\epsilon u : u \in A_m\}$, $0 < \epsilon < 1$, then $\gamma(\epsilon A_m) = \gamma(A_m) = m$. For $v = \epsilon u \in \epsilon A_m$, we have

$$\begin{aligned}\Phi(v) &\leq \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\epsilon^{p(x)}dx - \int_\Omega \frac{1}{s(x)}|u|^{s(x)}\epsilon^{s(x)}dx + C\int_\Omega (|u|^{\alpha_0}\epsilon^{\alpha_0} + |u|^{q^+}\epsilon^{q^+})dx \\ &\leq \frac{1}{p^-}\epsilon^{p^-} + C\epsilon^{q^+} - \frac{\delta_1}{s^+}\epsilon^{s^+} + C\delta_2\epsilon^{\alpha_0}.\end{aligned}$$

Without loss of generality, we may assume that $s^+ < q^+$ (see Remark 4.1. Since $2 \leq s^+ < p^-$ and $s^+ < \alpha_0$. For ϵ small enough, we have $\Phi(v) < 0$ for any $v \in \epsilon A_m$. Since $\Phi(0) = 0$ and the function $t \to \Phi(u(t))$ is nonincreasing, we have $\omega(\epsilon A_m) \subset \mathscr{A} \setminus \{0\}$. By Lemma 4.2 and the properties of the Z_2 index, we obtain that

$$\gamma(\mathscr{A} \setminus \mathcal{O}(0)) \geq m,$$

for some small neighborhood $\mathcal{O}(0)$ of 0. This completes the proof of the claim.

Finally, in the Mañé projection theorem we may take a linear (and thus odd) projection. Then every symmetric closed subset of the attractor (not containing zero) has a Z_2 index less than $2n + 1$ if the fractal dimension of $\mathscr{A}$ is less than n. Thus, the claim above implies that the fractal dimension of the global attractor is infinite. □

Remark 4.1. Actually, the assumption $s^+ < q^+$ is not necessary. From assumption (1.4), $F(x,u)$ can always be controlled by $C(|u|^{\alpha_0} + |u|^{\alpha_1})$ for some $\alpha_1 > \max\{q^+, s^+\}$. Thus, one can set V_m to be an m-dimensional subspace of $W_0^{1,p^+}(\Omega) \cap L^{\alpha_1}(\Omega)$, and set A_m to be the unit sphere in V_m.

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